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KANT'S DOCTRINE OF THE BASIS OF MATHEMATICS[1]

THE treatment of the philosophy of mathematics by Kant, in the 'Critique' and in the 'Prolegomena,' is equally characteristic of his philosophy in general, and of the age in which he did his work. The age in question was one of a rapid development in certain relatively advanced regions of mathematical research. But it was also an age of disillusionment regarding the power of mathematical science to demonstrate metaphysical truth. It was furthermore a time when mathematical inventiveness was decidedly more noticeable than mathematical rigor; and when construction had for the moment outrun logical reflection in mathematics. On the other hand, it was a time when the philosophers had learned many lessons concerning the importance of experience for their own constructions. Consequently it was a period when mathematics and philosophy were further apart, in spirit and in interest, than they had been during a portion of the seventeenth century. The mathematicians were in a sense more disposed to novel speculation and researches. The philosophers were less confident of the success of *a priori* constructions. Since the death of Leibnitz no thorough-going effort towards a philosophy of mathematics had been made. The efforts of Leibnitz himself regarding this topic were very imperfectly known in the age when Kant wrote. In brief, it was a moment when a sharp sundering of the task of the mathematician and of the philosopher appeared especially called for. That a critical philosopher should lay stress upon the contrast was, therefore, extremely natural. Even as a rationalist Kant had to feel that reason in philosophy had other offices than it had in mathematics. And the mathematicians of the time were too little possessed of an insight into the philosophy of their own science to give him any aid in bridging the chasm that seemed to him to divide the two kinds of activity. Furthermore, the way in which his own critical thought came to him, namely, through the reflections which first culminated in the year 1769, was a way which served to emphasize the contrast be-

[1] Read at the meeting of the American Philosophical Association, Philadelphia, December 28–30, 1904.

tween mathematical and philosophical truth. The discovery of the essay of 1769, on 'The Reason for the Distinction between Regions in Space,' was such as very soon to lead Kant to the conclusion which characterizes the dissertation of 1770, namely, to the conclusion that space and time predetermine the form of the phenomenal world of our percepts, but do not throw any light upon the ultimate nature of things. When Kant passed from the doctrine of the dissertation to the later deduction of the categories, he indeed learned to abandon all methods of obtaining knowledge of the noumena; but the failure to win this knowledge did not bring philosophy any nearer again to the position of mathematics. The business of philosophy remains for Kant the criticism of fundamental concepts, and the determination of their range of validity. The business of mathematics he conceived as the construction of those objects whose laws are determined by the forms of our perceptual faculty. This contrast of the two is henceforth, therefore, absolute. Philosophy, as the *Methodenlehre* of the 'Critique' explains to us, possesses no axiom of a theoretical character, and can justify its concepts only by an explicit proof of their necessity as the conditions of a possible experience. Philosophy, therefore, can never construct its objects of synthetic knowledge. Such objects, constructed and presented by the mind for itself and to itself, are the topics of mathematical science alone. The certainty of mathematical science depends entirely upon the necessity which for us our forms of perception possess. Kant never falters in his assurance that these, our forms of perception, are determinate, are finished, are for us absolutely predetermined by our constitution. In the 'Prolegomena' he triumphantly shows how the possibility of an exact knowledge of mathematical truth is explicable upon his theory and not upon any other. In emphasizing the contrast between mathematical and philosophical method, he expressly does so on the ground that no form of pure thinking can ever present to the mind an object, or can ever demonstrate the properties of this object otherwise than by a mere analysis of a prevously given concept. And Kant always confidently speaks as if mere analysis must necessarily lead to comparatively barren and unprogressive scientific precedure. By pure thought I can discover that man is rational, only in case my definition of man has already included rationality amongst the marks which are to be characteristic of man. If mathematical science is able to know objects *a priori* and in ways which are both synthetic and instructive, that must be because mathematical science depends upon something quite different from pure thinking. As this something is *a priori* and necessary, it can only be, in Kant's opinion, a form of perception, and our power to construct objects in accord-

ance with this form. So much, then, for the view of mathematics which Kant took.

The development of mathematical science since the time of Kant has followed a path which his influence has no doubt affected, but whose direction he was entirely unable to foresee. The mathematicians of Kant's time did indeed make unhesitating use of generalizations derived from the observations of objects constructed in space; and they made this use in a way which rigorous mathematicians no longer regard as justifiable. The mathematicians since the time of Kant have tended more and more to follow the very direction which he would have warned them not to follow. Namely, they have, on the whole, increasingly forsaken the method of trusting to perceptual construction as a means of mathematical demonstration. Geometry without diagrams is now the order of the day amongst the most rigorous students of the bases of geometry. Where diagrams are introduced, the reader is especially warned (as in Hilbert's recent lectures on the foundations of geometry, autographed for his hearers)—the reader is expressly warned, I say, to take as it were no logical notice of the diagrams, to regard them merely as hints, illustrations, suggestions of a relational structure whose consequences are to be developed without any use of the perceived properties of diagrams. In this sense it would seem as if the ideal of modern mathematics were the ideal of a science of pure concepts—the very ideal that Kant expressly declared to be impossible for mathematical science. Kant warned the philosophers that they must not attempt to use the methods of mathematics, just because they could not construct their concepts *a priori*. The modern mathematician is warned that he must not put his trust in the properties of visible figures, just because the ideal of his science, the ideal of the search for necessary conclusions, is an ideal which perceptual intuition rather confuses than directly furthers. As Kant interprets the business of mathematics, the mathematician has seen, and therefore believes. He believes because he has seen *a priori*. The modern logicians of mathematics would rather seem to say, Blessed is he who has not seen, in Kant's sense of the *reine Anschauung*, but has yet learned rationally to believe; for he alone has learned with true rigidity to grasp the meaning of his fundamental concepts.

As an incident of this whole development of modern mathematical logic there have appeared various doctrines concerning the bases of geometry which appear to be remote enough from those which Kant explicitly recognized. The tridimensionality of space is for Kant a result of the *a priori* form of intuition. The modern geometer would in general admit that we can indeed see no con-

ceptual reason why space must be limited to three dimensions. But instead of saying like Kant, that the limitation of space to three dimensions is something *a priori*, necessary and certain, the modern geometer would regard this limitation as something which, from the point of view of pure mathematics, is not necessary at all. The properties of a tridimensional space can be, with sufficient definition of the other properties of space, rationally developed. But the form of a tridimensional space is, logically speaking, only one of countless possible forms, whose logically definable properties are precisely as justified a topic of pure mathematics as are the properties of the space of our ordinary geometry. If you reply that tridimensional space is alone worthy to be called space, because that is the only kind of space that we happen to have, then the modern mathematician replies that this limitation may be as important as you please for philosophy, but is an empirical limitation, which makes tridimensional geometry of great importance for applied mathematics, but which, just because of the limitation, has nothing whatever that is mathematically necessary about it. In brief, show me a form of intuition of the Kantian type—so the modern logician of mathematics might say,—and if I accepted your account of it, I should regard it merely as a character belonging to a specifically defined human experience, a character which for that very reason would have no sort of mathematical necessity about it, and, therefore, no authority which need limit in the least mathematical generalizations which may be suggested by this form, but which may vary from it in any given way.

So much for tridimensionality. But of decidedly greater importance for modern theoretical geometry than the merely formal possibility of doing away with the limitation of geometry to three dimensions is that study of geometrical implications, necessities, and possibilities which has appeared in connection with the non-Euclidean geometry, and which still continues, in the form of constantly new additions to our present list of possible geometries. The geometry which Euclid found it convenient to work out explained the relations present in a large number of observable diagrams, by means of certain simple principles. As a fact, this explanation of the observed phenomena by the assumed principle was, in Euclid's case, incomplete, since there are demonstrations in Euclid which do not follow from the axioms alone, but which depend upon the observation of special diagrams. The modern geometer regards such demonstrations as unsatisfactory, just because they make use of principles which the diagram more or less unconsciously suggests, and which the Greek geometer did not make explicit in his list of axioms and postulates. In other words, that very use of intuition

which Kant regarded as geometrically ideal, the modern geometer regards as scientifically defective, because surreptitious. No mathematical exactness without explicit proof from assumed principles —such is the motto of the modern geometer. But suppose the reasoning of Euclid purified of this comparatively surreptitious appeal to intuition. Suppose that the principles of geometry are made quite explicit at the outset of the treatise, as Pieri and Hilbert or Professor Halstead or Dr. Veblen makes his principles explicit in his recent treatment of geometry. Then, indeed, geometry becomes for the modern mathematician a purely rational science, so far as any one special form of geometry is concerned. But hereupon a question of great philosophical interest becomes only all the more insistent. Any one form of geometry, such, for instance, as the Euclidean geometry, depends upon assuming the simultaneous truth of a number of distinct fundamental principles. It is possible to show, and in recent mathematical treatises it has been distinctly shown, that such principles can be so stated as to be logically quite independent of one another, so that no one of them could be deduced from the others. A system of objects which should conform to some of these principles and not to the others is therefore perfectly definable, is mathematically possible. This has been indubitably shown in case of the system of principles assumed by Euclid, and is all the more obvious when to Euclid's explicit principles are added such statements as make explicit the meaning of those principles which, guided by surreptitious appeals to intuition, he more or less unconsciously assumed. Under these circumstances, that very indifference to what we perceptually find present in this or in that diagram, that indifference, I say, which modern mathematical method encourages makes all the more inevitable the question: What necessity is there of assuming precisely that system of mutually independent first principles which Euclid found it convenient to assume, and which, with some supplements, the modern expositors of Euclidean geometry employ? Since it is demonstrable that no sort of logical inconsistency would be involved in supposing the existence of systems of objects which satisfy some of these principles and not others, whence, if from anywhere, is derived the authority of this particular system? As is well known, the modern logicians of mathematics differ a good deal in the theories of knowledge which they use in their answer to this question. But it may be said that very few students of the logic of mathematics at the present time can see any warrant in the analysis of geometrical truth for regarding just the Euclidean system of principles as possessing any discoverable necessity. The facts of the world of experience seem to be economically describable, so many say, in the terms of Euclidean

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geometry. But in this sense Euclidean geometry differs in no whit from the concepts of the theory of energy. Mathematical necessity belongs to the deductions from the principles, and to them only. For those who take this view a considerable range of difference of opinion still remains open, regarding the sense in which this convenience of Euclidean geometry as a means of describing the world is forced upon us by experience. Some are disposed to say, No other system of geometry seems to be probably applicable to our physical world. Some would insist that, for reasons upon which I need not here dwell, the known phenomena might be characterized in non-Euclidean terms, if only we could agree to accept certain conventions which actually run counter to our present mental habits. There are some mathematical logicians who are disposed to accept the Kantian view far enough to admit that the Euclidean space form is the expression of what we men, so long as we remain true to our present perceptual nature, must needs find the most natural way of interpreting spatial experience. But about one matter nearly all the modern students who approach the subject without a distinct pre-existing Kantian bias are agreed, namely, that whatever necessity belongs to Euclidean geometry, apart from the necessity of its deductions, is in no sense mathematical necessity, any more than the present necessity that bitter tastes, if sufficiently strong, should be disagreeable is a mathematical necessity. With this view I myself agree. It determines our judgment as to the positive value of Kant's view of the basis of mathematics. For mathematics, from the modern point of view, is *concerned with necessary inferences.* If the field within which necessary inference is itself a possible matter is in any way a restricted field, that is, if there are some subjects that admit of systematic mathematical treatment, while other subjects altogether forbid such treatment, then the field within which systems of exact mathematical inference are possible is determined by the categories of thought, and not by the forms of any intuition. The ideal of mathematical science is the exact development of the consequences of all those ideal forms which it is possible to subject to exact treatment at all.

Thus, if the *ego* has a determinate relational structure, and if this relational structure admits of mathematical treatment, then, and just in so far, the science of the *ego* will become a branch of mathematics. What will make it so, if at all, will not be the necessity under which we now stand of appreciating the presence of the *ego,* but the capacity which our concepts of the *ego* may possess of development in terms of a precisely definable system of relationships. In the same way, if our spatial experience presents a character which admits, as it does, of precise relational definition and development,

then we shall have, as we have, a mathematical geometry. But the mathematical necessity of this geometry will belong solely to the field where the exact development of the relational structure of the ideal entity called, for instance, Euclidean space, is possible. Mathematical necessity will in no wise be possessed by this entity itself as distinct from any other entity (say a non-Euclidean space), which can be treated with equal exactness. God may have made our space-perceiving nature on the lines of Euclid's geometry. If he did, that is a matter of experience, not of mathematical necessity.

If your boots have a relationally exact structure, there may be a mathematical science of this structure, precisely so far as the relations in question are exact. But it will be no part of mathematical science to determine whether or why you have any boots at all. If you insist that the form of your boots is determined *a priori* by the form of your feet (a proposition which may be regarded rather as advisable than as necessary), then the form of your feet will be a topic for mathematical science, precisely so far as the relations involved in this form constitute a system reducible to certain fundamental principles, and such that the characters of this system can be deduced from these principles. But mathematical science will have nothing to do with the question why you have any feet at all, or why you have not fins instead. If one conceives an absolute being possessed of a totality of perfect mathematical knowledge, so that it defined with absolute adequacy all possible relational systems, even such a being, so far as it was merely mathematical, would define only general types of ideal objects, and not individual objects such as *these* boots and feet and fins, unless, indeed, it added to its mathematical determination such will-decisions as distinguish the individual deed that we do from the possibility that we leave undone. In brief, a form of intuition, if such exists, is precisely a character belonging to the individual nature of man as a real being, and is not a mathematical necessity. It is a mathematical necessity that an ideal entity defined in general as 'a spendthrift' will become bankrupt *if* his capital is so much and *if* he regularly exceeds his income by so much a year. You can provisionally predict when such a spendthrift will become bankrupt. But there is no mathematical necessity that in the real world anybody should be a spendthrift. That result, if it happens, is due to free will, or to inherited disposition, or to training, or to the devil, or to whatever other existence you decide to take into account. As I regard this distinction between the general definition of an ideal necessity and the individual decision of a will as valid from an idealistic point of view just as much as from an empiristic point of view, as I regard the Absolute as subject to this distinction quite as much as we are, just because

it is an absolute distinction, I should myself fully agree that a Kantian form of intuition, if you can prove its existence in our own nature, has absolutely no interest as the foundation of any mathematical science, except in so far as it may suggest to some mathematician the particular ideal topics upon which he finds it convenient to build up a mathematical theory.

So much for the way in which the whole modern mathematical development is distinctly opposed to the Kantian conception that something called a form of intuition, distinct from a conceptual system, is a necessary basis of mathematical investigation. But there is indeed quite another aspect of the Kantian doctrine to be considered. Kant, after defining with a natural, but to us no longer interesting, narrowness the business which he calls the mere analysis of pure concepts, decided very correctly that so barren an undertaking as declaring that a rational animal is rational could be of no service for enlarging our knowledge. He accordingly maintains, in the form of the famous distinction between synthetic and analytic judgments, that every significant science which truly enlarges our knowledge depends upon a genuinely constructive and synthetic process. He also very correctly pointed out that every productive type of reasoning depends upon its own sort of experience. Whoever reasons, unquestionably observes something. That is, whoever considers some ideal object, and yet enlarges his knowledge as he does so, gets this knowledge from actually observing what happens to his idea as he works over it. Now observing what happens to one's ideas as one works over them is indeed definable as a kind of rational perception. But the possibility of such rational perception exists quite as much when you are considering the idea of God, or Kant's favorite idea of the possibility of experience, as when you are observing facts of spatial experience, or your boots. There is indeed a great difference between observing an ideal process, and making a decision as to which one of two ideas, whose consequences you have ideally observed, you shall henceforth allow to be individuated as the deed that you choose. There is also a sharp difference between observing such an ideal process, and looking to see whether that which the natural world permits to exist does or does not accord with your ideas. In either case you are observing a process which expresses a purpose. But abstract ideal processes without final and individual decisions, are observed in a way which differs from the way in which decisive and individual facts are identified as actually or as probably real facts of existence. For the ideal processes, with whose consequences mathematical science is alone concerned, are universals in the abstract sense. What you observe as the consequences of such an abstract idea may or may not accord with what your own personal

decision, or the decision of the world-will permits to exist as individual fact. Hence observing a mathematical necessity is never the same as observing an individual existence. And for the same reason observing a mathematical necessity is never the same as observing what we call a phenomenon of nature. Nevertheless, observing a mathematical necessity is indeed observing a process of ideal construction, and its results. And every such process of ideal construction unquestionably has a form. This form is, however, not what Kant meant by a form of intuition as distinct from a form of thought, for what you observe when you observe mathematical truth is a precisely and abstractly definable and general *necessity,* which is neither the perception of a fact in the natural world, nor yet a final decision of your own will, nor yet a metaphysically individual thing; but which is precisely *the general way in which this idea has to express itself.* The principle that guides one in such observations is unquestionably what Kant meant by the principle of contradiction, when he called this principle the principle of analytical judgment. So far as an idea is defined as a type of action, a plan, a way of behavior, it necessarily implies whatever is such that the contradictory of this consequence would be opposed to the idea itself. Whenever you observe such implications you observe a system of truth which comes to you as the system of the consequences of certain ideal processes. Such an observation is, however, an observation of synthesis, quite as much as it is an observation of the truth of Kantian analytical judgments. And as a fact, the lesson of Kant's whole deduction of the categories is that analysis apart from synthesis is impossible, and *vice versa.* In consequence, mathematical truth is indeed truth relating to a system of possible experience. And the mathematician observes the structure of this system empirically. Only because what he observes is an abstract process of construction, not an individual phenomenon, the truth that he discovers is of abstractly universal application to all things, whatever they may prove to be—and if such there be—that conform to his ideal constructions. Mathematical insight is, then, not without experience, and, if you please to use the term, not without intuition. But the intuition is not of perceived diagrams, nor of the special conditions of human experience, but of the relational structure of an ideal system.

Mathematical science has nothing to say, for instance, as to whether either human beings or the inhabitants of Mars are necessarily forced to count. Mathematical science defines the eternal validity of numerical truth. This truth is true for us. It is also true for the inhabitants of Mars, *if* there are any such. We experience it to be true because we try an ideal experiment, and see that

what was true in this ideal case must needs be true of an infinity of other ideal cases, precisely because of the abstract nature of what we have observed. This which we have observed must be true for all beings and at all times and places, because the opposite would be contradictory. But mathematical science has nothing to say as to whether or no we, or the inhabitants of Mars, must be beings such as are able to perceive this truth. Kant, however, was quite wrong in supposing that the application of the principle of contradiction would give us an analysis only of commonplaces. He was quite right in supposing that whenever we think we engage in a constructive process and observe something. But what we observe when we think is, as the non-Euclidean geometers show, frequently so general that we can define vast numbers of objects that we never hope or even desire to perceive; precisely as a moral agent is capable of conceiving accomplished plans of action that he never hopes to carry out, and that in many cases he deliberately forbids himself to carry out. Yet every consideration of a plan of action is an ideal sort of acting, which simply does not carry itself out into the individual deed, but remains abstract and general.

It is a perfectly fair question to ask, What is the universal form of that abstract type of ideal experience upon which all reasoning processes depend? This, however, is the question, not of the Kantian forms of intuition, but of the categories. The forms of thought are unquestionably the forms of mathematical science. That is what the whole recent mathematical theory has made manifest. On the other hand, the immortal soul of the Kantian doctrine of the forms of intuition remains this, that thinking itself is a kind of experience, that true thinking is synthetic as well as analytic, is engaged in construction of a peculiar kind, and not in mere barren analyses such as the statements that all rational animals are rational. Kant was right that the novelties of mathematical science are due to the observation of the results of constructive processes. He was even right that the observation of a diagram, in so far as the diagram is simply the expression of an idea, may be an admirable guide in the thinking process. He was wrong in supposing that a special form of intuition, such as that of Euclidean space, can have any other necessity than that which every individual fact in the world possesses. Every fact, in my opinion, is what some will decides it to be. Every fact is individual. But that does not determine the range of ideal possibilities, nor the range of mathematical truth. For mathematical truth is concerned with the consequences of ideas in advance of, or apart from, the decision whether those ideas which are then taken as what I have elsewhere called internal meanings, are expressed in individual realities. Mathematical science is ab-

stract, and can, therefore, never define the whole truth. For the whole truth of things is always individual, and is never expressible in terms merely of abstraction, nor in terms of merely logical implication. On the other hand, as soon as you consider any individual fact, as, for instance, the fact that this man has this form of intuition, you consider what, if true, is no topic of mathematical science. I conclude, then, that Kant's theory of the basis of mathematics has been in one respect wholly abandoned, and properly so, by the modern logic of mathematics. In another respect, precisely in so far as Kant declared that constructive synthesis and observation of its ideal results are both necessary for mathematics, Kant was unquestionably right. And as nobody before him had so clearly seen this fact, and as the progress of mathematical logic since his time has been so profoundly influenced by his criticisms, we owe to him an enormous advance in our reflective insight in this field.

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SOME OUTSTANDING PROBLEMS FOR PHILOSOPHY

FOR a long space of time the domains of philosophy and mathematics were regarded less as intersecting spheres of interest than as adjacent fields separated by a kind of 'dead line' that no worker in either might venture to cross without risking the loss both of his identity and of the respect of his fellows. I once heard a distinguished mathematician say that the study of mathematics acts on the metaphysical instinct like sulphur on the itch. Undoubtedly that savant had a philosophy, but, like the Irishman's snake, he was unconscious of it, and he was the less tolerant on that account. On the other hand, the mathematician has not infrequently been compelled to forgive such disrespect as that of Sir Wm. Hamilton's on ground analogous to the good old Catholic principle of invincible ignorance. Happily, the tokens more and more abound that the uncanny day of such misunderstandings is rapidly passing away. It may return again, but not for some generations. A new era has begun that shall be distinguished by intellectual sympathy and cooperation, by increasing wholesomeness of scholarship. The indicia have reference to old records. They indicate the reestablishment of broken traditions of an older time when philosopher and mathematician were often united in a single personality. Such men as C. S. Peirce and Pearson and Mach and Couturat and Poincaré and Georg Cantor, exemplify clearly enough that the larger incarna-

tions are not impossible even in these crowded days of narrow specialization.

Nowhere else is the community of interest so vast and conspicuous as in the theory of assemblages. By the latter I mean the doctrine variously entitled *Mengenlehre* and *Mannigfaltigkeitslehre* by the Germans, *théorie des ensembles* by the French, and often referred to in English as the theory of manifolds or aggregates, or by other analogous designations. Many of its ideas date at least as ancient as historical thought, and have figured in important ways in logic, in philosophy and in mathematics steadily from the earliest times. On the other hand, many of its chief concepts, its characteristic and ruling notions, and their organization into a distinct and self-supporting body of coherent doctrine, may be said to constitute the latest great mathematico-philosophical creation. If mathematics is the most fundamental of the sciences, assemblage theory is destined to be regarded as the most fundamental branch of mathematics. Herein lies its immeasurable import for philosophy. Viewed in retrospect, it appears as an inevitable product of the modern critical spirit. Already it is seen to underlie and interpenetrate both geometry and analysis. Its connection with modern logic is most intimate, often approximating identity with the latter. And philosophy in some wide-awake quarters has already recognized in the doctrine of manifolds her own most promising and inviting field.

Now in this young, vigorous and rapidly unfolding doctrine, there have been for some time certain recognized outstanding questions that ought to challenge mathematician and philosopher alike, in equal measure certainly, if not in manner the same. These problems are of two classes: those that have been very recently solved, and those which still await solution. I purpose to state some of the problems here in the hope that some rising philosopher may feel their challenge and, in case of the latter class, direct a yet unconquered genius to their solution.

A question of the former class is: *Can every assemblage be well-ordered?* It has just received an affirmative answer by E. Zermelo.[1] The fundamental character of the problem, the simplicity of the means employed in its solution and especially the bearings of the solution on kindred dependent questions, justify a brief report upon the matter in this place.

What, then, is an assemblage? Any collection of objects or things (elements) of whatsoever kind or kinds is an assemblage. An assemblage, to be mathematically available, to be available, that is, for the purposes of rigorous thought, must be *defined;* and it is

[1] *Mathematische Annalen*, Band 59, Heft 4, November, 1904.

said to be defined when and only when we know enough about it to know that an arbitrarily given object either is or is not an element of it. When is a defined assemblage said to be well-ordered (*wohlgeordnet, bien defini*)? The answer is, when and only when its elements have been so disposed in fact or in thought that one may affirm of it the following three propositions: (1) of any two of its elements *a* and *b*, one, as *a, precedes, i. e.,* is of *lower rank,* and the other, as *b, comes after, i. e.,* is of *higher rank;* (2) of any triplet of its elements *a, b, c,* if *a* is of lower rank than *b,* and *b* is lower than *c,* then *a* is lower than *c;* (3) the assemblage has an element of *lowest* rank, a *first* element, and the same is true of *every part* of the assemblage; that is, of every assemblage whose elements are elements of the given assemblage.

The simplest possible example of a well-ordered assemblage is that of the integers in their natural order 1, 2, 3, $\cdots$, ν, $\nu+1$, $\cdots$, it being agreed that any *element* (figure) ν shall be of *lower* rank than an element $\nu+\sigma$ if the *number* ν is *less* than the *number* $\nu+\sigma$. It is easily seen that the assemblage in question satisfies the three necessary and sufficient conditions. The same is true of the assemblage of all *rational* numbers greater than *zero* and less than 1, if it be agreed that of every two such numbers $\frac{a}{b}$ and $\frac{c}{d}$, that one shall have lower rank which has the smaller number for sum of its terms, and that, if $a+b=c+d$ in case of two numbers $\frac{a}{b}$ and $\frac{c}{d}$, then the smaller of these shall have the lower rank. Thus arranged, the assemblage stands: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{3}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{2}{5}$, $\frac{3}{4}$, $\cdots$. But if the same elements be arranged in their so-called natural order, that is, on a line of unit-length from zero to 1, and if it be agreed that of two numbers that one shall be of lower rank whose distance from the origin (zero) is the less, then, though the assemblage is indeed *ordered,* it is *not well-ordered,* for in its present arrangement it fails to satisfy condition (3); it has, for example, no *first* element, no element of lowest rank, and the like is true of countless numbers of its parts. It is plain, then, that not every assemblage, not even every ordered assemblage, is well-ordered. What Zermelo has established is that every assemblage *can be* well-ordered.

To such as know that it is impossible to *denumerate* the points of a continuum, that is, to set up a one-to-one correspondence between, say, the points of a unit-line and the integers, Zermelo's result is apt to be surprising, and, because of its far-reaching implications, it is really astonishing. Of such implications, two of the most important and striking are these: *first,* every conceivable assemblage can be so arranged that each element (unless it be the last, an insig-

nificant special case) shall be followed by a *next;* and *second,* that every assemblage can be so arranged that *every* sequence of its elements *a, b, c,* ··· in which *a* has higher rank than *b, b* higher rank than *c,* etc., must have an *end,* a last element. Any one who will seriously attempt to conceive the points of the room in which he is sitting so arranged that the assemblage shall have the two properties just stated, can not fail to have a deepening sense of the marvelous character, the awful comprehensiveness, of Zermelo's proposition.

And what, pray, is the stuff of which the demonstration is woven? What are the auxiliary concepts employed? Few indeed and simple, though tenuous and slippery and lithe. First is the notion of the assemblage of *all* the *parts* of a given assemblage E. Another idea is that of associating or pairing with each of those parts some definite element of the part, to be then known as the 'distinguished' (*ausgezeichnete*) element of its associate part. A third concept is that of the *off-cut* of a well-ordered assemblage, the term denoting the part (of a well-ordered assemblage) whose elements, all of them, are of lower rank than a given element of the assemblage. Obviously to each element of the assemblage there corresponds a definite off-cut. A very important rôle is played by the notion of *γ-assemblage:* an assemblage A is a γ-assemblage A provided it is well-ordered and, in addition, possesses the property that, if e is an arbitrary element of A and if O is the off-cut corresponding to e, then e is always the 'distinguished' element of $E - O$, where the latter symbol denotes all the elements of E except those of O. Such are the *Sommerfäden* out of which the author has contrived the beautiful figure of his demonstration.

I have already pointed out two of its important bearings. Another of its consequences is that it immediately removes from the field of doubt or controversy another question of the utmost fundamental significance. I refer to the question concerning the *comparability* of every pair of assemblages. The meaning of the question can be made sufficiently clear by a word of explanation. Two assemblages, A and B, are said to be *equivalent* when and only when it is possible to set up between the elements of A and those of B a one-to-one correspondence. The simplest familiar example of such equivalence is that where A denotes the assemblage of integers and B that of their doubles. Such examples abound in the worlds of space and number. On the other hand, A is said to be *less* than B and B greater than A, when A and B satisfy the two conditions: (1) A has no (proper)[2] part equivalent to B; (2) B has

[2] A proper part of A is a part composed of some but not all of the elements of A.

a (proper) part equivalent to A. Now either of the relations, equivalence and less (greater), excludes the other. That admits of rigorous demonstration. But herewith is not resolved the question whether one or other of the relations subsists between every two assemblages. If A and B are equivalent, then neither is less than the other; if one of them is less than the other, the two are not equivalent; but herein is plainly no ground for inference that they must be *either* equivalent *or* one less than the other. Possibly in some cases neither relation holds, *i. e.*, A and B may not be comparable. In such an eventuality, we should have to recognize two classes of assemblages: the comparable and the non-comparable. Behold the matter from another point of view (due to Borel). If A and B be two assemblages, then the following four cases may arise: (1) A has a part equivalent to B but B none equivalent to A; (2) B has a part equivalent to A but A none equivalent to B; (3) A has a part equivalent to B and B has a part equivalent to A; (4) A has no part equivalent to B and B none equivalent to A. The symmetry here is noteworthy. In passing, I wish to signalize the importance of the principle of symmetry in fundamental questions of logic, and to suggest the topic, the logical significance of esthetic principles, or its equivalent, as possibly worthy of an aspirant to the doctorate. To resume, it is noteworthy that the four cases are not merely hypothetical, they *exist*. Cases (1) and (2) are, of course, essentially the same, and are exemplified by taking for A or B a point continuum and for B or A the totality of integers. In such case one is less than the other. To exemplify case (3) it suffices to let A stand for the assemblage of real (rational and irrational) numbers and to let B denote the assemblage of the irrational numbers. It has been proved that if A and B fall under case (3), they are equivalent. Case (4) alone remains. That it is not *empty* is shown at once by letting A denote the assemblage a_1, a_2 and B the assemblage b_1, b_2. In this *particular* example, A and B are indeed equivalent, but does the notion of equivalence attach to *every* pair A and B of (4), a case shown to exist? Georg Cantor, easily the primate of all contributors to *Mengenlehre*, believed the answer should be affirmative. But he had not been able to give a scientific, *i. e.*, a *transferable*,[3] demonstration, and men of the type of Borel held the matter in doubt. Was Cantor's conviction *rooted* in esthetic soil? Was it due to a fine commanding sense of what esthetically ought to be? At all events his conviction was just, if not logically justified, for it is an obvious corollary to Zermelo's

[3] Concepts and proofs are essentially *social* affairs. They must be intelligible to at least two minds, or, what is tantamount, to one person at least twice. Such need not be true of emotion.

proposition that every two assemblages are comparable and that, if they belong to case (4), they are equivalent.

So much for 'problems' that may be regarded as very recently settled. I turn now to the second kind. Of these the most conspicuous is that of determining whether there is thinkable an assemblage having a power intermediate to that of the denumberable assemblage and that of the continuum. These terms demand explanation. If the assemblages A and B are *equivalent*, they are said to have the *same power* (*Mächtigkeit*). An assemblage having the same power as the assemblage $a_1, a_2, a_3, \cdots, a_\nu, \cdots$, without end, is said to be denumerable, or to have the power of the denumerable assemblage. To avoid confusion, it is well to note that *power* has not been defined; it is *sameness of power* that has been defined. An assemblage, such as that of the irrational numbers, that has the same power as that of the real numbers, or the points of a line-segment, is said to have the *power of the continuum*. Now it is well known and readily admits of proof that, if A has the power of the denumerable assemblage and B the power of the continuum, A and B have *not* the same power. In fact, as above noted, A is, in the case supposed, less than B. Is there an assemblage I intermediate to such an A and B? That means, is there an I greater than A and less than B? All efforts to prove or disprove that such is the case have been thus far baffled. The proof offered by the late Paul Tannery has been found defective. The problem meets one on the very threshold of assemblage theory and *apparently* demands acumen and ingenuity rather than a large fund of mathematical knowledge.

It may be useful to view the question from another point of view. As is well known, Cantor has defined 'classes' (I), (II), $\cdots$ of higher and higher power, where by *higher* is here meant that (I) regarded as an assemblage is less than (II) similarly regarded, and so on. This achievement of his, which is one of the boldest and most brilliant in the annals of mathematic genius, was done as follows. He made use of three *principles of number generation*. These, stated in full generality, are: (1) the addition of unity, 1, to a number already formed; (2) positing, in case of an endless sequence of numbers (integers) having no greatest, a new number that shall be the first integer after the sequence and immediately greater than every number of the sequence; (3) subjecting all numbers generable by (1) and (2) to the condition that all those preceding any specified one of them shall have the same power as a class already defined.

Starting with 1, class (I), $1, 2, \cdots, \nu, \cdots$, is defined by (1). By (2), ω is put to be first after (I) and greater than every number of (I). Combination of (1) and (2) yields $\omega + 1, \omega + 2, \cdots, 2\omega,$

$2\omega + 2, \cdots, \omega^2, \omega^2 + 1, \cdots, n\omega^2, \cdots, n\omega^\omega, \cdots$. Class (II) is struck out by application of (3) to the foregoing sequence of transfinites. Cantor proves that (I) and (II) are not of the same power and that there is no assemblage between (I) and (II) in the sense of intermediate above explained. Hence our problem is to show that (II) has the power of the continuum or that it has not. It is interesting to note that with exquisite impropriety Cantor has denominated his three principles *Logical Moments*. Moments they undoubtedly are; but logical, they are just as undoubtedly not. Neither are they illogical. They are super-logical, being precisely the critical points in his process of transfinite creation where logic pauses for a *new start*, the adoption of a *principium*, a choice, never compelled, but only solicited and always declinable. The possibility, the spiritual vision, of a generalization, logic may and often does present, but the deed of generalization logic is impotent to do or force.

I shall close by stating another outstanding question that is immediately suggested by the procedure just described, though the same question is encountered on many another path. It is: Is there in the *constitution* of the human mind any limitation to the definition of higher and higher powers of assemblages? Is, in other words, an assemblage of all assemblages and elements logically available? Is the notion meaningless? Is it free from interior contradiction? Would such an assemblage have to be an element of (in) itself? Whatever the answers be, can we, do we, ever or always, dispense with such a 'notion'? Can the word *all* be safely used *conjunctively*—in the sense of none excluded?

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REVIEWS AND ABSTRACTS OF LITERATURE

Elements of Metaphysics. A. E. TAYLOR. London, Methuen & Co.; New York, The Macmillan Co. 1903. Pp. xvi + 419.

This book is the most comprehensive and systematic treatise on metaphysics that we have yet had from a living writer in English. It discusses with definiteness and clearness of style and in well-ordered arrangement the chief problems of ultimate philosophy. Professor Taylor is a disciple, though not a slavish follower, of Mr. Bradley, and he endeavors in a constructive and original fashion to unite the general speculative position of Mr. Bradley with the interpretations of actual experience which have found fullest expression and elucidation in the writings of Professors Ward and Royce. And whatever one's opinion as to the final coherence of the various strains in Professor Taylor's doctrine one must

recognize the eminent skill with which he has illustrated and defended his position in relation to the physical and psychological sciences. In fact I should be disposed to say that the herein attempted synthesis of Bradleian absolutism with the teleological conception of knowledge and the self is most instructive in the books devoted to 'Cosmology and Rational Psychology.'

The first book, entitled 'General Notions,' discusses successively the 'Problem of the Metaphysician,' 'Metaphysical Criterion and Metaphysical Method' and the 'Subdivisions of Metaphysics.' Metaphysics, we are told, deals with the *general* character of reality, and, while by its intellectual or reflective procedure it differs from poetry and religion, its goal is the same as that of mysticism. The method of metaphysics is *analytical, critical, non-empirical* and *non-inductive;* its material is *immediate experience, 'psychical content'* or *'psychical matter of fact';* and its criterion, the law of contradiction. Reality is identified at the outset with immediate experience or psychical content in so far as this is self-consistent (p. 20). And here at the outset Professor Taylor prejudges the whole case by an offhand assertion that reality must be self-consistent. We may say that *knowledge* of reality must be self-consistent, but is there any meaning in making such an assertion about reality if reality be identical with experience? There seems to be a confusion in the assertion that experience is bound to conform to the law of contradiction. For it is precisely the contradictions in experience that drive us beyond itself to a *thought-construction* of reality. Again, if we base philosophical construction on actual experience, is it not misleading to assert that metaphysical procedure is *non-empirical* and *non-inductive*? This, it seems to me, would be to commit philosophy to a barren formalism; and, indeed, Professor Taylor's own work is perhaps most valuable when he does not stick to his method.

In the second book, on the 'General Structure of Reality,' the author lays down the fundamental principles of his metaphysics. In chapter I., 'Reality and Experience,' the absolute or ultimate reality is defined as that structure of the world-system of which all purposes, each in its own way, must take account. The author has already defined reality as psychical content or *immediate* experience, and he now endeavors to show that these definitions coincide. Purpose springs from, and seeks its goal in, *feeling.* 'To say that reality is essentially one with immediate feeling is only another way of saying that the real is essentially that which is of significance for the attainment of purpose' (p. 55) (to which, it seems to me, should be added 'and for the *defeat* of purpose'). But purpose is always the expression of *individual interest,* and so Professor Taylor adds to his definitions this: 'Reality is uniquely individual.' Throughout the whole work constant use is made of this teleological conception of reality, and insistent emphasis is put on the fact that Reality is immediate psychical content or *feeling,* in which the distinctions of reflective cognitions have been merged and lost, oneness of feeling and insight (p. 61), etc. Now this higher and integral immediate experience he sometimes recognizes as including the labor of thought as this is developed

through ideal activity; but he nowhere makes clear the relation of thought to experience. After all it is individual selves who *have* experience and in whom thought is organic to experience. It is surely no more legitimate to take some type of feeling-experience, such as esthetic or personal emotion, separate it from its place in the total life of the self, and hypostatize it as the Absolute, than it would be to take pure thought. What is this but *Erfahrung überhaupt*, an abstraction like *Bewusstsein überhaupt?* And, while the author repudiates the category of whole and part as inadequate to the nature of reality, he frequently speaks of the 'absolute' or 'universe' as one whole of experience, which at once includes, but in itself differs from, the finite individual experiences which are its contents. It is just here that difficulties begin with Professor Taylor's doctrine. In the remaining chapters of Book II., 'The Systematic Unity of Reality,' 'Reality and its Appearance—the Degrees of Reality,' 'Substance, Quality and Relation,' and 'Change and Causality,' we find discussed the main characteristics of the 'Absolute' or 'Ultimate Reality' in relation to the world of finite individuals.

The degree of reality which anything possesses, we are told, corresponds to its degree of *individuality*, and the degree of 'individuality' depends (1) on *comprehensiveness;* (2) on *inherent systematization'* (p. 110). The whole of Reality is the one perfect individual (p. 113), and its individuality means that it is the systematic embodiment of a single coherent structure in a plurality of elements or parts, which depend for their whole character upon the fact that they are the embodiments of precisely this structure. So far each finite individual seems to have his integral place and function in the whole. But now we find the emphasis falling on the *unity, continuity* and *timelessness* of Reality. The whole of reality is one infinite substance (p. 139), emphasis is laid on its supra-relational character, and it is argued in the criticism of causation that the view that there is discontinuity in events is inherently self-contradictory. (The force of the argument depends here on ruling out the possibility that causal interaction is an immediate relationship of real self-active, and, therefore, so far *discrete* centers of being.) These and similar expressions reveal the Spinozistic trend of Professor Taylor's argument, and the treatment of *time, evolution* and the *self* in Books III, and IV. becomes significant in this regard. The author recognizes that time is essentially bound up with the life of finite selves, and is hence *par excellence* the historical category. And, in his valuable discussion of evolution, he admits a *proximate* reality in the process of evolution which is evaluated in terms of finite individuality. Furthermore, the reality of the self is bound up with growth or development. But the whole of Reality can not develop; therefore evolution, history and selves with their discontinuous series of actions and experiences (p. 311) have no place in Ultimate Reality. For the entire time-process is appearance, and the lives of finite selves are bound up with it.

Now, on the other hand, we are told in the concluding chapter that the Absolute is the final realization of our intellectual and our practical ideals. And great stress is laid throughout on the *teleological* character of indi-

viduality in science, in action and in experience, as a key to the nature of reality. Now it seems to me that human *interest, purpose, striving,* as marks of spiritual individuality, are essentially matters of growth, of history, of what our author calls the time-process. The timeless, supra-relational Absolute Experience and the finite, teleological activity of the human individual lie unreconciled in his system. He starts out from the Bradleian Absolute in which all relations and distinctions are submerged or absorbed, and then he recognizes a relative reality in the actual dynamic and historical movement of selves; but he does not achieve a synthesis of these two *motifs.* He does not, indeed, face at all the question, what must ultimately be meant by ascribing any reality to the finite as a 'content' of the Absolute Experience. The latter is above all the distinctions and imperfections of human experience. How then do these enter into and become constitutive of its placid, static unity? How do their pains contribute to its eternal bliss? We are told, indeed, in the chapter on the Self that the absolute unity of experience may be best prefigured as a *society of selves.* But in any actual society the unity is effective and the society coheres just in so far as the unity is a principle of action in each and every individual self. But the absolute is no self, and since all selves are finite the infinite unity of experience can not dwell in their thoughts alone. Where then is it? It seems to me that a more consistent consideration of the actual nature of so-called finite selves, and a closer regard of the time-process, would have led the author farther away from his Spinozistic and Bradleian Absolute, and closer to an interpretation of reality in terms of selfhood, and to a higher valuation of the time-process. It would have led, too, to a recognition of the inadequacy of such conceptions as psychical 'content' or 'matter of fact' to describe reality, in view of the purposive or self-active character of the human individual. And finally it would involve a more explicit recognition of the transcendence of experience which is involved in metaphysics.

In Book III., 'Cosmology,' the author works out in a very interesting and convincing manner the theory that the physical order must be the presentation to our sense of a system or complex of systems of sentient subjects, and that the mechanical view of nature has only the value of a methodological postulate or working view for physical science. Hence the principles of mechanical physics have only relative validity. Hence, too, 'laws of nature' represent only statistical averages, and we have no guarantee that actual concrete cases exhibit *exact* conformity to law. "Our failure to detect specific forms of sentience and purpose in what we commonly call 'inorganic' nature need mean no more than that we are here dealing with types of experience too remote from our own for detection" (p. 209). In refuting the mechanical view of nature Professor Taylor neatly shows that the very conception of a machine arises from and involves intelligence and purpose.

The whole treatment of 'Matter,' 'Law,' 'Descriptive Science,' etc., constitutes one of the very best bits of cosmological discussion in English.

In chapter I. of Book IV. the author argues effectively against the purely atomistic conception of psychology as a surrogate for an incom-

plete physiology. He recognizes that the psychological conception of conscious life is a 'transformation' of actual experience, and explains this transformation in terms of Avenarius's 'introjection' theory. Professor Taylor finds that psychology has the positive and important function of furnishing general symbols for ethical and historical appreciation. In the next chapter he argues for the interactionist view as the best scientific working hypothesis as to the relation of soul and body, laying stress on the breakdown of parallelism in the face of the teleological aspects of mental life, and its consequent failure to make room for the application of psychology to ethical and historical appreciations. The author makes it clear that the whole problem is a 'scientific' one, inasmuch as immediate experience reveals no dualism of body and soul.

I have already referred to his treatment of the 'self' in the next chapter. There are many points that might be discussed here, but space limits forbid my referring to more than one. Professor Taylor argues that, since the self requires an external environment and depends on the contrast-effect generated by social relations, the self is essentially finite. It seems to me that the argument equally applies to 'experience.' The latter involves the contrast between the experiencing 'subject' and the 'content' or 'object' of experience. And it is a mere quibble to say that we have some experiences without this distinction, since in knowing and asserting that we have them we are already conscious of the distinction. I think a good case could be made out for an absolute self by developing the conception of environment as 'organic,' rather than merely external to the self, and of 'social contrast' as a phase of self-expression and intercommunion.

The chapter on 'Moral Freedom' is one of the best in the book. Here the author shows the common ground of the fallacies of determinism and indeterminism in the assumption that all rational connection must be of the type embodied in the temporal causal series, and points out that we do not know and can not say that a man's character is fixed as thus and so until he has actually expressed his individuality; and that hence, although a man's action is the expression of his selfhood, we can never predict absolutely what he will do. Is, then, character created in and by action?

In chapter V, 'Some Metaphysical Implications of Ethics and Religion,' it is argued that Reality must somehow make provision for the gratification of our ethical, religious, and esthetic interests. But we cannot say *how* this is done, and 'perfect virtue,' 'perfect happiness' and 'infinite progress' are self-contradictory concepts. In religion we conceive of the ideal of perfection as already existing in an individual form. God is the timeless, infinite Absolute, and in him evil is seen to be mere illusory appearance. Here again it seems to me when we recur to the Absolute our finite interests, activities and deeds come perilously near losing all their reality, and our old difficulty as to the relation of the Absolute to the teleologically conceived human self becomes very dubious.

In the last chapter we are told that the Absolute transcends both thought and will. The conclusion involves an element of agnosticism

and of mysticism. Metaphysics adds nothing to our information, and yields no fresh springs of action.

The work is provided with a splendid analytical table of contents. Whatever one's personal attitude toward the theories advanced, one must pay tribute to the wide knowledge, speculative keenness and clearness of expression of Professor Taylor's work. No serious student of fundamental problems can afford to neglect this book.

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Time and Reality. JOHN E. BOODIN. *Monograph Supplement, Psychological Review,* October, 1904. Vol. VI., No. 3. Pp. 119.

The common opinion is that time is one-dimensional, continuous and something that essentially involves direction. These views Professor Boodin contests. He identifies time with the mere 'fleetingness' of reality; he would make its function exclusively negative. It is 'absolute non-being.' 'Relative non-being has to do with differences at different points of reality,' but absolute non-being is 'dynamic,' it is 'that property of the real subject-object' (*i. e.,* of the real world) which 'creeps into all our systems of truth and falsifies them, necessitating new ones,' which makes necessary 'incompatible judgments,' that is, 'different judgments as regards the same attribute of reality at the same point' (p. 28).

The question raised by this definition is distinct. It is not whether of several uses of the term time one shall be selected by convention as *the* use. We are familiar with the 'paradoxes' of time, due to the supposition that one term, and presumably one thing, includes aspects of reality in some way antithetical. Dr. Boodin would make the import of time perfectly simple, and, indeed, *purely negative.* And the question therefore is, what is this absolute non-being? And what is the significance of calling it time?

Absolute non-being is our *invention* 'to account for passing away and novelty' (p. 118). It is the antithesis to the 'permanent or habit' character of reality. 'Precisely the opposite of what is said of space must be said of time' (p. 29). But in every description that Dr. Boodin gives of time, except in this bare definition of it, we find included phases of permanence and positivity. For example, time, in necessitating incompatible judgments, must hold to that identity to which the incompatibility relates. Indeed, time alone makes possible any kind of judgment (p. 62). It no less necessitates new systems of truth than falsifies old ones (p. 28). Admittedly (p. 30), without the relatively stable space system we could have no measurement of time process, and time as a negative property would be inconceivable. But what makes our 'invention' conceivable is surely *part of* its power 'to account for passing away' (v. s.), and is therefore part of that 'invention.' And how could pure negativity be measured?

Does not Dr. Boodin's account of time as pure negativity thus break down entirely? If, however, we would name the positive qualities of time, we must meet his objection that the properties we would most readily

predicate of time themselves presuppose time, that to call time continuous, serial or quantitative, *e. g.*, is 'meaningless,' 'begging the question' or 'viciously circular.' The logic of definition rests under a cloud, I believe. So I would consider Dr. Boodin's application of the same principle of criticism in cases where the issue is simpler.

For example, he rejects as circular the Kantian definition of continuity (pp. 42–3). Kant said we have continuity when between two points, however selected, a third can always be found. Dr. Boodin takes this to mean that the points themselves make the continuum, and notes that they do not in the case of the rational fractions. He concludes that Kant gave so 'futile' a definition of continuity only because he had unawares begged the question by presupposing a continuum. I suppose one must answer that Kant meant merely to give the mark by which we may know that points are *in* (not *are*) a continuum; so that such 'circularity' is the definition's best commendation. Just as in the geometer's definition of a point or a philosopher's definition of substance. Aristotle said a continuum is found where the parts have common boundaries. But what glues the boundaries together, asks Dr. Boodin. A continuum, by definition, exists where the boundaries are common and need no gluing. If this is circular it is not vicious.

I conclude that to be thus circular *may* commend the description of time as serial, continuous, etc. I would also point out that to define time as absolute or *dynamic* non-being is 'circular' in the extreme. For dynamic, the significant term, as well as such parallel descriptions as 'creeping in,' 'necessitating,' and 'falsifying,' are in terms of process and imply time, the thing defined. But such and all criticisms of his definition are more or less clouded by a doubt. For Professor Boodin distinguishes between the world of reality and the world of truth. In the latter, time is our invention to 'account for passing away'; but *really*, time is indescribable, it is only the positive that is described. So perhaps all criticisms apply only to the 'invention,' not to the real, which may be as Dr. Boodin describes it. And yet it can not be described!

Ostensibly Dr. Boodin regards the category of potentiality as but obscurely plausible (p. 73); purpose in the world means our subjective purposes, not valid of the real (p. 85). Science is right in discarding teleological categories for purposes of explanation (p. 71). All explanation, so far as it attains its ideal, is of the type of mechanical causation, where the effect equals the cause and nothing really happens (p. 48–52). Hence Dr. Boodin describes the world as dual, a world of process on the one hand, a world of truth, a 'static stare,' on the other. All description must be timeless (p. 62). The self in seeking to unify experience ignores the very process it *means* to explain (p. 52). Time renders all our systems of truth invalid; yet we have (p. 119) and should have (p. 52) a faith that 'truth shall abide.'

One may ask whether this is 'really' so, or only 'truly' so. If to the world of 'truth and ideal construction' one opposes the world of the real and indescribable process, the elusive 'individual core of being' (p. 63), then we must admit that the very duality here stated, and the difference,

is 'ideal construction.' It is, then, but a 'shadow' of the real. And, if our doubt here is self-destructive, what must be said of a faith in the abiding of a truth which is never true—of anything, and never persists through time at all.

The assumption here is that process can not be described. Dr. Boodin in his very statement of this position describes the process by which the result is attained. The categories of potentiality and purpose are his readiest tools; tendency and implication and fulfillment and direction are the fruits time brings him with the seeds of growth and purpose. Does not the biologist or the historian, too, describe life and development as *such,* in terms that we accept as descriptive of the real, or as failing in that, not because they eliminate process, but in contrast with another process known and described?

Surely, then, if we are to say anything at all of time we can not deny to it at least *direction.* Even the static 'now' of immediate experience, through which things flow, gives to the flow a 'before and after' character in virtue of a similar distinction we find attached to that 'now' *as now.* And this direction character is yet more obvious in the sense in which time flows. But with this direction comes in at least the potentiality of series and quantity; for between before and after is implied at least a third somewhat.

Simultaneity, moreover, is not simply banished from time, but the temporality is bound up with the assertion that the existence of *b* requires the non-existence of *a* and *c,* because *a, b, c* form a certain kind of series, and are therefore simultaneous in another sense. This serial character we find also *in* the 'now,' giving it at least two bounds and something between. Thus *b,* the now, appears successively in perhaps many different lights, all essential to its elementary time character; and series, continuity and quantity seem successively to be denied and asserted of time, in different connections.

These remarks I suppose to be entirely trite. They are but preliminary to an expression of wonder that the attempt should be made, as it is in Dr. Boodin's book and in Hobhouse's 'Theory of Knowledge,' to present time as something simple, or, as in Bradley's well-known work, as something contradictory. Both attempts seem to flow from the idea that the *concept* time must be a perfectly simple thing, like—well, I know not what. Surely the task is to fill out the account, such as Aristotle offered, of the way in which various conceptions are held together in the one conception of time, and not, after the entirely false pattern of generalization, to seek one abstract concept, other than time, which shall 'unify' by obliterating all the complexity of time.

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JOURNALS AND NEW BOOKS

ARCHIV FÜR SYSTEMATISCHE PHILOSOPHIE. Band X., Heft 4, December, 1904. *Die Aufgabe wissenschaftlicher Asthetik* (pp. 433–480): C. D. PFLAUM. – Scientific esthetics is defined as the knowledge of the purely intensive valuations of the contents of the soul. In support, the opinions are cited of authorities from Plato to Jonas Cohn, the definition being taken from the last. Particularly it is maintained that Kant, in spite of appearances, did not really deny the possibility of a scientific esthetic, nor exclude therefrom consideration of values. *Über die Verwechslung des sinnlich Angenehmen mit den Kunsteindrucken und einige andere Folgen der sogenannten empirischen Ästhetik* (pp. 481–509): R. SKALA. – *Naturgefühl* and *Kunstgefühl* are to be identified. Empirical esthetics considers esthetic valuation only from the subjective side. Fechner makes the distinction between pleasant and esthetic feelings one of degree; but, in fact, he overlooks the esthetic feeling altogether. Empirical esthetics is responsible for the modern idea that two arts taken together will produce more effect than either one alone. *Gerechtigkeit* (pp. 510–520): B. STERN. – Righteousness is, first, to avoid interference in the psychical life of other men and of beasts, and, second, in that of one's own psychical life. *Das Problem der Willensfreiheit vom Standpunkt des Sollens* (pp. 521–540): H. STAEPS. – Freedom rests on belief in the power of the ideal. It frees us from the dominion of natural lusts and from egoism; it gives rule to the latent force of truth, beauty and goodness. *Jahresbericht über Erscheinungen der Soziologie in den Jahren 1799–1904* (pp. 543–561): R. GOLDSCHIED. – Special attention is paid to '*Altersklassen und Männerbünde. Eine Darstellung der Grundformen der Gesellschaft*' by H. Schurtz.

ARCHIV DER GESCHICHTE DER PHILOSOPHIE. Band XI., Heft 2. January, 1905. *Die Atomistik und Faradays Begriff der Materie* (pp. 139–166): O. BUEK. – Faraday's theory is contrasted with that of Boskowich. The latter presents definite logical difficulties; the former is incomplete, being the basis of Thompson's recent theories. Fechner's error lay in taking nature as something given instead of a useful construct. Through this error have crept in mythological and childish treatments, such as Zöllner's. The logical relation of Faraday's doctrine to that of Helmholtz and other more modern theories is still to be determined. *Voltaire als Philosoph* (pp. 166–215): P. SAKMANN. – A detailed study of Voltaire's conception of philosophy, his epistemology, his cosmology, his idea of God and proof of God's existence. While he followed Locke in the main, his philosophical capacity was greater than is commonly supposed. *Herder und Tetens* (pp. 216–249): W. UBELE. – The comparison refers to their treatment of the theory of language. Tetens is less rich, but more precise; he emphasized the practical origin of speech and the place of conception therein. Herder asserted speech

to be fundamentally a matter of emotion. *Le commentaire arabe d'Averroës sur quelques petits écrits physiques d'Aristote* (pp. 150–252): H. DERENBOURG. *Vétilles d'un lecteur de Platon* (pp. 253–264): L. M. BILLIA. – A discussion of texts and interpretations in three of the dialogues. *Die 'Geschichte der Philosophie' am zweiten philosophischen Kongress in Genf* (4–8 September, 1904) (pp. 265–270): K. JUNGMANN. – *Jahresbericht; Die Polnische Philosophie der Letzten zehn Jahre* (1894–1904): H. VON STRUVE. – *Die neuesten Erscheinungen. Eingegangene Bücher.*

VIERTELJAHRSCHRIFT FÜR WISSENSCHAFTLICHE PHILOSOPHIE UND SOZIOLOGIE. Band XXVIII., Heft 4. December, 1904. *Uber ein Paradoxon in der Logik Bolzanos* (pp. 375–392): J. K. KREIBIG. – Bolzano contended that in some instances the law of the inverse proportions between the content and extent of a concept does not hold; his examples are not true exceptions, but serve to bring out certain neglected features of the law. *Die Geschichte der Erziehung in soziologischer Beleuchtung* (pp. 393–421): P. BARTH. – The growth of the 'encyclopedic education' in Greece, and its social causes; similar social changes in Rome lead to the borrowing of Greek educational ideals. Individualism the fundamental cause of degeneracy. *Zu Kants und Lockes Gedachtnis* (pp. 422–426): P. BARTH. *Besprechungen* (pp. 427–472): K. Lamprecht, *Deutsche Geschichte:* P. BARTH. F. Mauthner, *Beiträge zu einer Kritik der Sprache:* P. BARTH. S. Witasek, *Grundzüge der allgemeinen Asthetik:* H. SPITZER. A. Posada, *Literatura y Problemas de la Sociologia:* E. DI CARLO. E. Oscar, *Nietzsches Lehre, in ihren Grundbegriffen:* R. RICHTER. H. Cohen, *System der Philosophie:* I. H. GRÜNBAUM. *Berichtigung und Erwiderung. Philosophische Zeitschriften. Bibliographie.*

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Frenzel, Karl, *Ueber die Grundlagen der exacten Naturwissenchaften.* Leipzig und Wien: Franz Denticke. 1905. 8vo. Pp. 145. 3 M.

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NOTES AND NEWS

THE group of experimental psychologists which met at Cornell University during the last Easter recess, held its second meeting at Clark University on Friday, March 31, and Saturday, April 1, Professor E. C. Sanford presiding. The programme was begun on Friday afternoon by an inspection of the Clark Laboratory, during which three reports were made by advanced students on research work in progress in the laboratory. Mr. L. M. Terman discussed 'Tests of Bright and Dull Boys,' Mr. A. L. Gesell, 'The Correlation of Handwriting and School Standing,' Mr. W. F. Bock, 'The Psychology of Acquisition as Shown in Learning to Typewrite.' The first session was opened by Professor A. H. Pierce, who reported on researches now under way in the Smith College Laboratory on the effects of imperceptible shadows. Discussion by Professors Titchener and Sanford followed. Mr. Stevens, of Cornell University, next read a paper by Professor Max Meyer on 'Auditory Sensation in an Elementary Course,' which was discussed by Professors Titchener, Sanford and Pierce. On Friday evening, Professor Sanford gave an informal dinner to the guests in attendance. The Saturday morning session was addressed by President G. Stanley Hall, who read a paper on 'Tendencies and Dangers in Experimental Psychology,' which was followed by an interesting discussion, led by Professor E. B. Titchener. Dr. J. P. Hylan followed, and Professor Titchener and President Hall again spoke by way of reply to criticisms. After the discussion the psychologists repaired in a body to the physical laboratory where Professor Arthur G. Webster gave a demonstration of a new apparatus for the measurement of the absolute intensity of tone. President and Mrs. Hall received the members for luncheon at the President's house, after which the closing meeting was held to hear a paper by Professor I. Madison Bentley on 'Tonal Analysis,' followed by a demonstration in the laboratory. After a vote of thanks to the authorities of Clark University and in particular to President Hall, Professor Sanford and Librarian Wilson, the meeting adjourned and the guests were given an opportunity of inspecting the laboratories and the library of the university. It was agreed that the next meeting should be held in the spring of 1906 either at the Yale laboratory or at the University of Pennsylvania.

THE Section of Anthropology and Psychology of the New York Academy of Sciences in conjunction with the New York Section of the American Psychological Association met on March 27, 1905, at the Psychological Laboratory of Yale University in response to an invitation of the Philosophical Club and the Anthropological Club of Yale. The psychological laboratory and the anthropological collection at the Peabody Museum were open to visitors throughout the day. There were two sessions, afternoon and evening, and the visitors dined with their hosts at the Graduate Club. The following papers were read: 'Central Anæsthesia During Eye Movement,' Raymond Dodge. 'Movements of

Convergence,' Charles H. Judd. 'Radical Differences in the Upper Limit of Audibility,' Frank G. Bruner. 'Variations in Sung Tones,' E. H. Cameron. 'Perception of Linguistic Sounds,' F. L. Wells. 'Mental Growth in Deficient Children,' Naomi Norsworthy. 'Classification of the Senses,' Robert MacDougall. 'Memory Tests of Faces,' Will S. Monroe. 'Transference of Practice,' G. Cutler Fracker. 'Practice and Training,' J. McKeen Cattell. 'Studies in Reading Aloud,' L. A. Weigle. 'Chance,' W. L. Sheldon. 'Types of Monism,' W. P. Montague.

THE Science Hall of Denison University, containing the laboratories of chemistry, geology and biology and the scientific library, was destroyed by fire on the morning of March 30. The files of the *Journal of Comparative Neurology and Psychology* and many valuable books and papers were lost.

DR. ROBERT M. YERKES, of the department of psychology of Harvard University, has been awarded the Boyleston Medical Prize for 1905, the subject of his essay being, 'Auditory-Tactual Reinforcement and Inhibition in the Frog.'

GEORGE V. N. DEARBORN, Ph.D., M.D. (Columbia), has been promoted to the full professorship of physiology in the Tufts College Medical and Dental Schools.

PROFESSOR WILHELM OSTWALD, the eminent physical chemist of Leipzig, will again this year take part in the work of the summer school of the University of California. He has been appointed also on the Harvard faculty under the arrangement recently made by Harvard for an exchange of professors with German universities, and will lecture at Harvard for a half-year.

PROFESSOR WILLIAM JAMES sailed for Greece on March 11, to be gone until June. He will attend the International Congress of Psychology at Rome.

DR. WILLIAM P. MONTAGUE, tutor in philosophy in Columbia University, has been appointed instructor in the same university.